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Linear Chaos and Approximation

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Let $\{T_n(t)\}_{n=1}^{\infty}$ be a sequence of strongly continuous linear semigroups on Banach spaces X_n converging in the sense of Kato to a semigroup $T(t)$ on the Banach space X. We discuss under what conditions the chaoticity of $T_n(t)$ is inherited by $T(t)$. We apply our results to a discrete parabolic equation. © 2000 Academic Press

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1. INTRODUCTION

During the past 20 years or so, several definitions of chaos have been proposed (see [Dev, K-S, L-Y, Wig]). One of the most used among these definitions can be formulated as follows [Dev]:

Definition 1.1. Let X be a metric space. A continuous map $f: X \mapsto X$ is said to be *chaotic* on X, if

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(1) f is topologically transitive;

(2) the periodic points of f are dense in X;

(3) f has sensitive dependence on initial conditions.

Condition (1) means that for all nonempty open subsets U and V of X there exists a natural number n such that $f^n(U) \cap V \neq \emptyset$, while a point x is called *periodic* if there exists some $n \in \mathbb{N}$, $n > 1$ such that $f^n(x) = x$. For the notion of *sensitive dependence* on initial conditions we refer to [Dev, Definition 8.2, p. 49]. In the following we shall dispense with condition (3). Indeed, it has been proven in [Ba-al] that if f is transitive and has dense periodic points then f has sensitive dependence on initial conditions. One of the most appealing properties of Definition 1.1 is that it extends rather naturally to linear system ([McC, P-A, D-S-W]). More precisely, let X be an infinite-dimensional separable Banach space and T a linear bounded operator on X. The operator T is called *hypercyclic* if there exists $x^* \in X$ such that its orbit under the operator T, $\{T^n x^* \mid n \in \mathbb{N}\}$, is dense in X. The operator T is hypercyclic if and only if it is topologically transitive (see, for instance, [D-S-W, Theorem 2.2]). Thus, a hypercyclic operator T is chaotic if and only if the set of X_{per} of the periodic points of T, i.e.,

$$X_{\text{per}} := \{x \in X \mid T^n x = x, \text{ for some } n \in \mathbb{N}\},$$

is dense in X. Let us now restate Definition 1.1 in the context of linear semigroups defined on a Banach space.

DEFINITION 1.2. A C_0-semigroup $\{T(t)\}_{t \geqslant 0}$ on the Banach space X is called *chaotic* provided that $T(t)$ satisfies the following two conditions:

(H) (Hypercyclicity) There exists $x^* \in X$ such that $\{T(t)\, x^* \mid t \geqslant 0\}$ is dense in X.

(P) The set X_{per} of the periodic points of $\{T(t)\}_{t \geqslant 0}$ given by $X_{\text{per}} := \{x \in X \mid \exists t > 0 : T(t)\, x = x\}$ is dense in X.

Our objective in this paper is to investigate the chaoticity of the approximating semigroups. In order to analyze this question, we define the approximation in the sense of Kato, which is one of the most general frameworks in approximation theory for linear semigroups (see [Kat, Chap. IX, Sect. 4]). Let $(X_n, \|\cdot\|_n)$ be a sequence of Banach spaces such that on each X_n there is defined a C_0-semigroup $T_n(t)$. Assume that $T_n(t)$ converges in the sense of Kato to a C_0-semigroup $T(t)$, i.e.,

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \|T_n(t)\, P_n f - P_n T(t)\, f\|_n = 0 \qquad \text{for any} \quad f \in X, \quad (1.1)$$

for some approximating operator P_n (the precise definition is given in Section 3).

To approximate a chaotic semigroup $T(t)$ by a sequence of C_0-semigroups $\{T_n(t)\}$, one should first settle the following questions:

(Q1) If $\{T_n(t)\}$ is a sequence of linear chaotic C_0-semigroups which converges to $T(t)$ in the sense of Kato, can we assert that $T(t)$ is chaotic?

(Q2) Assuming that $T(t)$ is chaotic and $\{T_n(t)\}$ converges to $T(t)$ in the sense of Kato, under which conditions can we assert that at least one of $T_n(t)$ is chaotic?

The relevance of these questions can be seen in the two examples of the next section. The first example is constructed to show that in general the statement

(S1) $\qquad T_n(t)$ is chaotic $\forall n \in \mathbb{N} \Rightarrow T(t)$ is chaotic

is not true, which gives a negative answer to question (Q1). The converse statement;

(S2) $\qquad T(t)$ is chaotic $\Rightarrow T_n(t)$ is chaotic $\forall n \in \mathbb{N}$,

is not true either. But we can show that (S2) is true for those n's such that instead of (1.1) we have

$$P_n T(t) f = T_n(t) P_n f \qquad \forall f \in X. \tag{1.2}$$

We apply the above result to the chaotic semigroup generated by a parabolic equation which will be introduced in Example 2.4 of the following section. We do not expect that $e^{tA_h} \xrightarrow{K} e^{tA}$ (as is mentioned in [Kat], even for the heat semigroup we cannot expect such convergence). In Section 4 we construct explicitly the approximating spaces and we prove that for some specific mesh sizes the discretized problem becomes chaotic. We mention that this feature has already been observed and studied in the context of nonlinear chaos (see [T-W-P]).

2. TWO EXAMPLES

In this section we give two examples of linear chaotic semigroups.

EXAMPLE 2.1 [P-A]. Le $X := \ell^1$ and $A := -\alpha I + \beta B$, where B is the backward shift in ℓ^1; i.e., if $\mathbf{f} = (f_1, f_2, ...) \in \ell^1$, then $(B\mathbf{f})_n = f_{n+1}$. Then the linear evolution problem in X,

$$\begin{cases} \dfrac{d\mathbf{f}}{dt} = -\alpha I\mathbf{f} + \beta B\mathbf{f} := A\mathbf{f}, \\ \mathbf{f}(0) = \mathbf{f}_0 \in \ell^1, \end{cases}$$

generates a chaotic semigroup $T(t) = e^{tA}$, provided that $\beta > \alpha \geqslant 0$.

For the reader's convenience we briefly sketch the proof by using a lemma which gives a sufficient condition that a C_0-semigroup be hypercyclic. For the original proof see [P-A].

LEMMA 2.2 (see [D-S-W, Theorem 2.3]). *Let $\{T(t)\}_{t\geqslant 0}$ be a C_0-semigroup on the Banach space X. Define*

$$X_0 := \{x \in X \mid \lim_{t\to\infty} T(t)\, x = 0\}$$

and $X_\infty := \{x \in X \mid$ for any $\varepsilon > 0$, there exist $u \in X$ and $t > 0$ such that $\|u\| < \varepsilon$ and $\|T(t)\, u - x\| < \varepsilon\}$. If X_0 and X_∞ are both dense in X, then $\{T(t)\}_{t\geqslant 0}$ is hypercyclic.

Thus, in order for $\{T(t)\}_{t\geqslant 0}$ to be a chaotic semigroup in X, it suffices that X_0, X_∞, and X_{per} be simultaneously dense in X.

Proof of Example 2.1. The spectrum of A is $\sigma(A) = \{-\alpha + \beta\mu \mid |\mu| \leqslant 1\}$. Indeed for any $|\mu| < 1$, the vector

$$\mathbf{h}_\mu = \{\mu^n\}_{n\in\mathbf{N}}$$

satisfies the eigenvalue equation $A\mathbf{h} = \gamma\mathbf{h}$ with eigenvalue $\gamma = -\alpha + \beta\mu$ and the points with $|\mu| = 1$ belong to the spectrum by the closure property.

We assume that $0 \leqslant \frac{\alpha}{\beta} < 1$.

(1) *X_0 is dense in X.* Indeed, for $0 \leqslant \mu < \frac{\alpha}{\beta}$, one has $e^{tA}\mathbf{h}_\mu = e^{-\alpha t + \beta\mu t}\mathbf{h}_\mu \to 0$ as $t \to \infty$. Hence $\mathbf{h}_\mu$ and any finite linear combination of these vectors belongs to X_0. Since this set of vectors is dense in X, X_0 is also dense in X.

(2) *X_∞ is dense in X.* Since e^{tA} is a C_0-group, for $1 > \mu > \frac{\alpha}{\beta}$, one has $e^{-tA}\mathbf{h}_\mu = e^{\alpha t - \beta\mu t}\mathbf{h}_\mu \to 0$ as $t \to \infty$. Any such $\mathbf{h}_\mu \in X_\infty$, since for any $\varepsilon > 0$ one can choose t large enough such that for $\mathbf{g}_\mu = e^{-tA}\mathbf{h}_\mu$, $\|\mathbf{g}_\mu\| < \varepsilon$ and $\|e^{tA}\mathbf{g}_\mu - \mathbf{h}_\mu\| = 0$. The same argument shows that X_∞ is dense in X.

(3) *X_{per} is dense in X.* Indeed, since $\beta > \alpha$, the spectrum of the operator A, $\sigma(A)$, contains a nonempty segment of the imaginary axis. On that segment, the complex numbers $\kappa = i\,\frac{m}{n}$, $(m, n) \in \mathbb{Z}^2$ form a dense set. Each such κ is an eigenvalue of A with the eigenvector $\mathbf{h}_{(\kappa+\alpha)/\beta} = (\frac{\kappa+\alpha}{\beta}, (\frac{\kappa+\alpha}{\beta})^2, ...)$. Each $\mathbf{h}_{(\kappa+\alpha)/\beta} \in X_{\text{per}}$ and linear combinations of

such vectors are dense in ℓ^1. Indeed, if they were not dense, there would exist a nonidentically zero functional $\Phi = (\phi_1, \phi_2, ...) \in \ell^\infty$ such that $\langle \Phi, \mathbf{h}_{(\kappa+\alpha)/\beta} \rangle = 0$ for any $\mathbf{h}_{(\kappa+\alpha)/\beta}$ constructed as above. This would imply that the analytic function represented by the series $\phi_1 \frac{\kappa+\alpha}{\beta} + \phi_2 (\frac{\kappa+\alpha}{\beta})^2 + \phi_3 (\frac{\kappa+\alpha}{\beta})^3 + \cdots$ is zero on a set of points κ with an accumulation point, which implies it is identically zero, in contradiction with the assumption that $\Phi \neq 0$. ∎

LEMMA 2.3. *If in the above example $\alpha = \beta$, then the generating C_0-semigroup e^{tA} is not chaotic.*

Proof. For $\alpha = \beta$, $-\alpha I + \beta B$ has no purely imaginary eigenvalues and the uniform boundedness

$$\|T(t)\| = \|e^{\alpha t(B-1)}\| = e^{-\alpha t} \; \|e^{t\alpha B}\| \leqslant e^{-\alpha t} e^{\|t\alpha B\|} = 1$$

contradicts the hypercyclicity of the semigroup, since it is now clear that the orbit of any vector will be contained in the unit ball of X. Thus $T(t)$ cannot be chaotic. ∎

EXAMPLE 2.4 [D-S-W, Example 4.12]. The linear evolution problem is a convection-diffusion type equation of the form:

$$\begin{cases} f_t = a f_{xx} + b f_x + c f := A f, \\ f(0, t) = 0 \qquad \text{for} \quad t \geqslant 0, \\ f(x, 0) = f_0(x) \qquad \text{for} \quad x \geqslant 0, \end{cases}$$

with some $f_0 \in L^1(\mathbb{R}^+, \mathbb{C})$. In [D-S-W], this problem is considered in the Hilbert space $L^2(\mathbb{R}_+, \mathbb{C})$ but the same proof is valid in the Banach space $L^1(\mathbb{R}^+, \mathbb{C})$. So the operator A, with domain $\{ f \in W^{2,1}([0, \infty)) \mid f(0) = 0 \}$, generates an analytic chaotic semigroup $T(t) = e^{tA}$ in $L^1(\mathbb{R}^+, \mathbb{C})$, provided that a, b, $c > 0$ and $c < b^2/(2a) < 1$.

If in the above example we replace df/dx by its finite difference approximation $[\delta f]_j = (f_j - f_{j-1})/h_n$ and $d^2 f/dx^2$ by

$$[\delta^2 f]_j = \frac{f_{j+1} - 2 f_j + f_{j-1}}{h_n^2},$$

then the generator A of Example 2.4 becomes

$$[Af]_j = a \frac{f_{j+1} - 2 f_j + f_{j-1}}{h_n^2} + b \frac{f_j - f_{j-1}}{h_n} + c f_j.$$

By taking $h_n = a/b$ one gets $[Af]_j = (-(b^2/a)+c)\, f_j + (b^2/a)\, f_{j+1}$ and this is the generator of Example 2.1, namely $Af := -\alpha I + \beta B$ with $\alpha = (b^2/a) - c$ and $\beta = b^2/a$. Since $c < b^2/2a$, this implies $0 < \alpha < \beta$, which is a sufficient condition for e^{tA} in Example 2.1 to be chaotic. By disregarding the fact that the underlying spaces are different in these examples, the condition $h_n = a/b$ answers the question (Q2). In Section 4 we construct the precise spaces to answer this question.

3. APPROXIMATION IN THE SENSE OF KATO

DEFINITION 3.1. We say that a sequence of Banach spaces $\{(X_n, \|\cdot\|_n)\colon n = 1, 2, ...\}$ *converges* to a Banach space $(X, \|\cdot\|)$ *in the sense of Kato* and we write

$$X_n \xrightarrow{K} X$$

if for any n there is a linear operator $P_n \in \mathscr{L}(X, X_n)$ (called an *approximating* operator satisfying the following two conditions:

(K1) $\lim_{n\to\infty} \|P_n f\|_n = \|f\|$ for any $f \in X$;

(K2) for any $f_n \in X_n$, there exists $f^{(n)} \in X$ such that $f_n = P_n f^{(n)}$ with $\|f^{(n)}\| \leqslant C \|f_n\|_n$ (C is independent of n).

DEFINITION 3.2. Let $X_n \xrightarrow{K} X$, $B_n \in \mathscr{L}(X_n)$, and $B \in \mathscr{L}(X)$. We say that B_n *converges* to B *in the sense of Kato* and we write $B_n \xrightarrow{K} B$ if $\lim_{n\to\infty} \|B_n P_n f - P_n B f\|_n = 0$ for any $f \in X$. Let A_n and A be the generators of the C_0-semigroups $\{T_n(t)\}_{t\geqslant 0} \subseteq \mathscr{L}(X_n)$ and $\{T(t)\}_{t\geqslant 0} \subseteq \mathscr{L}(X)$, respectively. Consider the following three conditions:

(A) (Consistency) There is a complex number λ contained in the resolvent sets $\bigcap_{n\in\mathbb{N}} \rho(A_n)$ and $\rho(A)$, respectively, such that

$$(\lambda - A_n)^{-1} \xrightarrow{K} (\lambda - A)^{-1}.$$

(B) (Stability) There exists a positive constant M and a real number ω such that

$$\|T_n(t)\| \leqslant M e^{\omega t}, \qquad \text{for any } t \geqslant 0 \quad \text{and for any } n \in \mathbb{N}.$$

(C) (Convergence) For any finite $T > 0$

$$T_n(t) \xrightarrow{K} T(t)$$

uniformly on $[0, T]$, i.e.

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \|T_n(t) P_n f - P_n T(t) f\|_n = 0 \qquad \text{for any } f \in X. \tag{3.1}$$

In [Ush] one can retrieve the standard version of the Lax equivalence theorem which says that the conditions (A) and (B) hold if and only if (C) holds.

The following theorem gives a negative answer to question (Q1).

THEOREM 3.3. *There exists a sequence of chaotic semigroups which converges in the sense of Kato to a nonchaotic semigroup.*

Proof. By using the semigroup of Example 2.1, one can construct such a sequence. Let $X := \ell^1$ and $A := -\alpha I + \beta B$. From Lemma 2.3 the C_0-semigroup $T(t) = e^{tA}$ is chaotic for $\beta > \alpha > 0$ and nonchaotic for $\beta = \alpha > 0$. Hence, if $\{\beta_n\}$ is a sequence such that $\beta_n > \alpha$ and $\beta_n \to \alpha$, then the sequence of chaotic semigroups $\{T_n(t)\}_{t \geq 0}$ converges uniformly to the semigroup $\{T(t)\}_{t \geq 0}$. By taking $X_n := X$ and $P_n := I$ the identity operator, the sequence $\{T_n(t)\}_{t \geq 0}$ converges also in the sense of Kato to $\{T(t)\}_{t \geq 0}$, which is not chaotic. ▮

THEOREM 3.4. *Let $\{(X_n, \|\cdot\|_n) \mid n = 1, 2, ...\}$ be a sequence of Banach spaces such that $X_n \xrightarrow{K} X$. Suppose $\{T(t)\}_{t \geq 0}$ is a chaotic semigroup on $(X, \|\cdot\|)$ and on each $(X_n, \|\cdot\|_n)$ one defines a C_0-semigroup $\{T_n(t)\}_{t \geq 0}$. Now, if for some $n \in \mathbb{N}$ one has*

$$P_n T(t) f = T_n(t) P_n f \qquad \textit{for any } f \in X \quad \textit{and} \quad \textit{for any } t \geq 0, \tag{3.2}$$

then $\{T_n(t)\}_{t \geq 0}$ is also chaotic.

Proof. From the hypercyclicity of $\{T(t)\}_{t \geq 0}$ it follows that there exists an $f^* \in X$ such that $\{T(t) f^* \mid t \geq 0\}$ is dense in X. To prove (H) for $\{T_n(t)\}_{t \geq 0}$, take $g_n \in X_n$; from (K2) there exists $g^{(n)} \in X$ such that $g_n = P_n g^{(n)}$. Then for any $\varepsilon > 0$ there exists $t > 0$ such that $\|g^{(n)} - T(t) f^*\| < \varepsilon$. The assumption (K1) implies the uniform boundedness of $\{P_n\}$; hence for $f_n^* := P_n f^*$ we have

$$\begin{aligned}\|g_n - T_n(t) f_n^*\|_n &= \|g_n - T_n(t) P_n f^*\|_n = \|P_n(g^{(n)} - T(t) f^*)\|_n \\ &\leq M \|g^{(n)} - T(t) f^*\| \leq M\varepsilon.\end{aligned}$$

To prove (P) for T_n, take $g_n \in X_n$ and denote $g^{(n)} \in X$ such that $g_n = P_n g^{(n)}$. For $\varepsilon > 0$ we take $f^{(p)} \in X_{\text{per}}$ such that $\|g^{(n)} - f^{(p)}\| < \varepsilon$, then for $f_n^{(p)} = P_n f^{(p)}$ we have

$$\|g_n - f_n^{(p)}\|_n = \|P_n(g^{(n)} - f^{(p)})\|_n \leqslant M\varepsilon.$$

To finish the proof it suffices to see that $f_n^{(p)}$ is a periodic vector for T_n. Indeed,

$$T_n(t)\, f_n^{(p)} = T_n(t)\, P_n f^{(p)} = P_n T(t)\, f^{(p)} = P_n f^{(p)} = f_n^{(p)}. \quad \blacksquare$$

To prove the converse we have to impose extra conditions on P_n. Let us denote by (X_n) per the set of all periodic vectors of T_n in X_n and for any constant $C > 0$ let us define

$$\mathscr{C}(f_n) := \{ f^{(n)} \in X \mid P_n f^{(n)} = f_n \text{ with } \|f^{(n)}\| \leqslant C\, \|f_n\|_n \}.$$

THEOREM 3.5. *Suppose that (3.2) holds for some $n \in \mathbb{N}$. Suppose $\{T_n(t)\}_{t \geqslant 0}$ is chaotic and P_n satisfies $(X_n)_{per} \subseteq P_n(X_{per})$. If there exists a constant C such that for every $f \in X$ and $\varepsilon > 0$ there is an $f^{(n)} \in \mathscr{C}(P_n f)$ with $\|f - f^{(n)}\| < \varepsilon$, then $\{T(t)\}_{t \geqslant 0}$ is also chaotic.*

Proof. Let us prove that condition (H) holds for $\{T(t)\}_{t \geqslant 0}$. Since $\{T_n(t)\}_{t \geqslant 0}$ is hypercyclic, there exists $f_n^* \in X_n$ such that the orbit $\{T_n(t)\, f_n^* \mid t \geqslant 0\}$ is dense in X_n. Let $g \in X$ and $g_n = P_n g$. For any $\varepsilon > 0$ there exists $t > 0$ such that $\|g_n - T_n(t)\, f_n^*\|_n < \varepsilon$. According to (K2), let $f_*^{(n)} \in X$ be such that $f_n^* = P_n f_*^{(n)}$. Then for $h = g - T(t)\, f_*^{(n)}$ there exists $h^{(n)} \in \mathscr{C}(P_n h)$, with $\|h - h^{(n)}\| < \varepsilon$ and $P_n h^{(n)} = P_n h = g_n - P_n T(t)\, f_*^{(n)}$. As a consequence of (3.2), we obtain that $g_n - P_n T(t)\, f_*^{(n)} = g_n - T_n(t) - f_n^*$. Thus the inequality

$$\begin{aligned} \|g - T(t)\, f_*^{(n)}\| &\leqslant \|h - h^{(n)}\| + \|h^{(n)}\| \\ &\leqslant \varepsilon + C\, \|g_n - T_n(t)\, f_n^*\|_n \\ &\leqslant (1 + C)\, \varepsilon \end{aligned} \tag{3.3}$$

implies the hypercyclicity of $T(t)$. To prove (P) for $T(t)$, take $g \in X$ and $g_n = P_n g$. For $\varepsilon > 0$ we choose $f_n^{(p)} \in (X_n)_{\text{per}}$ such that $\|g_n - f_n^{(p)}\|_n < \varepsilon$. The bijectivity of P_n on X_{per} implies that we may choose $f^{(p)} \in X_{\text{per}}$ such that $f_n^{(p)} = P_n f^{(p)}$. By taking $h = g - f^{(p)}$ and $h^{(n)} \in \mathscr{C}(P_n h)$, an inequality similar to (3.3) implies the theorem. $\blacksquare$

4. CHAOTICITY OF THE DISCRETE PARABOLIC EQUATION

In this section we construct approximating operators for $X := L^1([0, \infty))$ in the sense of Kato. To this end, we define for each $n \in \mathbb{N}$ the step size h_n such that $h_n \to 0$ as $n \to \infty$. The corresponding space would be the space of step functions generated by the characteristic functions $\chi_j^n := \chi_{[(j-1)h_n, jh_n)}$ of the intervals $[(j-1)h_n, jh_n)$. That is, any function $f_n \in X_n$ can be expressed by

$$f_n = \sum_{j=1}^{\infty} a_j^n \chi_j^n. \tag{4.1}$$

Each X_n is endowed with the norm induced by X. Hence, if $f_n \in X_n$ is given by (4.1), then $\|f_n\| := h_n \sum_{j=1}^{\infty} |a_j^n|$. Now we define P_n as a map from X into X_n by

$$f_n = P_n f = \sum_{j=1}^{\infty} f(jh_n)\, \chi_j^n.$$

Riemann's sum theorem implies that

$$\lim_{n \to \infty} \|P_n f\|_n = \lim_{h_n \to 0} h_n \sum_{j=1}^{\infty} |f(jh_n)| = \|f\|.$$

Hence we have (K1); for (K2) it suffices to remark that the injection of X_n into X is isometric. It follows that the operator P_n is an approximating operator for X.

Now let us consider $f \in X$, $f_n := P_n f$ and define the operator A_n by

$$A_n f_n := \beta \sum_{j=1}^{\infty} f((j+1)h_n)\, \chi_j^n - \alpha \sum_{j=1}^{\infty} f(jh_n)\, \chi_j^n,$$

with $D(A_n) = X_n$.

LEMMA 4.1. *For $\beta > \alpha \geqslant 0$ the operator A_n generates a chaotic semigroup, e^{tA_n}, on X_n.*

Proof. One can repeat the proof of Example 2.1, by taking $\mathbf{h}_\mu = \sum_{j=1}^{\infty} \mu^j \chi_j^n$ as the eigenfunction of A_n.

Here we want to define an operator A such that

$$P_n A f = A_n P_n f \qquad \text{for any} \quad f \in D(A). \tag{4.2}$$

Actually the operator A is not exactly $aD^2 + bD + cI$ of Example 2.4, but a discrete version thereof, namely

$$A_h := a\left(\left(\frac{1}{h}\chi_{h/2}\right) * D\right)^2 + b\left(\frac{1}{h}\chi_h * D\right) + cI, \tag{4.3}$$

where χ_h and $\chi_{h/2}$ are the characteristic functions of $[0, h)$ and $[-\frac{h}{2}, \frac{h}{2})$.

LEMMA 4.2. *For $h = h_n = a/b$, $\alpha = ((b^2/a) - c)$, and $\beta = b^2/a$ we have*

$$P_n A_h f = A_n P_n f. \tag{4.4}$$

Proof. We can write

$$\left[\frac{1}{h}\chi_h * D\right] f(x) = \frac{1}{h}\int_0^h f'(x-y)\,dy = \left[\frac{f(x) - f(x-h)}{h}\right] \tag{4.5}$$

and for $h = h_n$

$$P_n\left[\frac{1}{h}\chi_h * D\right] f(x) = \frac{1}{h}\sum_{j=1}^{\infty}\left[f(jh) - f((j-1)\,h)\right]\chi_j^n(x).$$

In the same manner

$$\left[\frac{1}{h}\chi_{\frac{h}{2}} * D\right]^2 f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{4.6}$$

and

$$P_n\left[\left(\frac{1}{h}\chi_{h/2} * D\right)^2\right] f(x)$$

$$= \frac{1}{h^2}\sum_{j=1}^{\infty}\left[f((j+1)\,h) - 2f(jh) + f((j-1)\,h)\right]\chi_j^n(x).$$

Thus, if $A_h := a((\frac{1}{h}\chi_{h/2}) * D)^2 + b(\frac{1}{h}\chi_h * D) + cI$, we obtain (4.4) provided that $h = a/b$, $\alpha = ((b^2/a) - c)$ and $\beta = b^2/a$. ∎

LEMMA 4.3. *Suppose that e^{tA} and e^{tA_n} are the C_0-semigroups generated by A and A_n, which are related by relation* (4.2). *Then*

$$P_n e^{tA} f = e^{tA_n} P_n f \qquad \text{for any} \quad f \in X. \tag{4.7}$$

Proof. It is sufficient to differentiate, at least for $f \in D(A)$:

$$\frac{d}{dt}[P_n e^{tA} f] = P_n \frac{d}{dt}[e^{tA} f] = P_n A e^{tA} f = A_n [P_n e^{tA} f],$$

for all $t \geqslant 0$, and identify

$$[P_n e^{tA} f]|_{t=0} = P_n f,$$

so that $[P_n e^{tA} f]$ and $e^{tA_n} P_n f$ are both the mild solutions of the Cauchy problem

$$\begin{cases} \dfrac{d}{dt} u = A_n u; \\ u(0) = P_n f. \end{cases}$$

Since $D(A)$ is dense, (4.7) is true for all $f \in X$. ∎

From (4.5) and (4.6) it follows that A_h is a bounded operator on $L^1([0, \infty))$. Consequently it generates a uniformly continuous semigroup e^{tA_h}. Furthermore, if f is a step function, then $A_h f$ is also a step function; hence the subspace X_n is invariant under A_h and e^{tA_h}. Then we can prove:

THEOREM 4.4. *For $h = a/b$, e^{tA_h} is a chaotic semigroup on $L^1([0, \infty))$.*

Proof. For the proof we will use Theorem 3.5. Lemma 4.3 implies (3.2) and Lemma 4.1 asserts that e^{tA_n} is chaotic. To see that P_n is a bijective mapping between X_{per} and $(X_n)_{\text{per}}$, one takes $f \in X_{\text{per}}$ and writes $e^{tA_n} P_n f = P_n e^{tA_h} f = P_n f$. Thus, $P_n f \in (X_n)_{\text{per}}$. Conversely, if $f_n \in (X_n)_{\text{per}}$ by considering f_n as an element of X (since X_n is a subset of X) we have

$$P_n e^{tA_h} f_n = e^{tA_n} P_n f_n = e^{tA_n} f_n = f_n,$$

since P_n acts as the identity on X_n. On the other hand, $f_n \in X_n$ implies $e^{tA_h} f_n \in X_n$; thus $P_n e^{tA_h} f_n = e^{tA_h} f_n$ and consequently $e^{tA_h} f_n = f_n$.

Finally, for given $f \in X$, any function $g \in L^1([0, \infty))$ which satisfies $|g| \leqslant |P_n f|$ and interpolates f on the points $\{(jh, f(jh)), j = 1, 2, \ldots\}$ belongs to $\mathscr{C}(P_n f)$ with $C = 1$. Thus for any $\varepsilon > 0$ we can find an interpolating function $f^{(n)}$ with the additional condition $\|f^{(n)} - f\| < \varepsilon$. This concludes the theorem. ∎

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REFERENCES

[Ba-al] J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, On Devaney's definition of chaos, *Amer. Math. Monthly* **99** (1992), 332–334.

[D-S-W] W. Desch, W. Schappacher, and G. F. Webb, Hypercyclic and chaotic semigroup of linear operators, *Ergodic Theory Dynam. System* **17** (1997), 793–819.

[Dev] R. L. Devaney, "An Introduction to Chaotic Dynamical Systems," Addison–Wesley, New York, 1989.

[G-S] G. Godefroy and J. H. Shapiro, Operators with dense, invariant cyclic vector manifolds, *J. Functional Anal.* **98** (1991), 229–269.

[Kat] T. Kato, "Perturbation Theory for Linear Operators," Springer-Verlag, New York/Berlin, 1966.

[K-S] U. Kirchgraber and D. Stoffer, On the definition of chaos, *Z. Angew. Math. Mech.* **69** (1989), 175–185.

[L-Y] T. Y. Li and J. A. Yorke, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.

[McC] C. R. MacCluer, Chaos in linear distributed systems, *J. Dynam. Systems, Measure. Control* **114** (1992), 322–324.

[P-A] V. Protopopescu and Y. Y. Azmy, Topological chaos for a class of linear models, *Math. Models Methods Appl. Sci.* **2** (1992), 79–90.

[T-W-P] E. Twizell, Y. Wang, and W. G. Price, Chaos-free numerical solution of reaction-diffusion equations, *Proc. Roy. Soc. London A* **430** (1990), 541–576.

[Ush] T. Ushijima, Approximation theory for semigroups of linear operators, *Japan J. Math.* **1** (1975), 185–224.

[Wig] S. Wiggins, "An Introduction to Applied Nonlinear Dynamical Systems and Chaos," Springer-Verlag, New York/Berlin, 1990.